Regular and irregular regimes of binary fluid convection excited by parametric resonance

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A set of two asymptotically correct amplitude equations for envelopes of two counterpropagating waves in a horizontal layer of a binary fluid, excited by parametric resonance, is derived and studied. The only stable stationary monochromatic solution possible is standing wave (SW). It can lose its stability in several ways. Sideband instability of SW, similar to that studied by Eckhaus, is possible here and results in either the onset of SW with different wavelength or an irregular regime. The transitions caused by sideband instability are studied numerically. A qualitative explanation of the irregular behavior is proposed. [S1063-651X(97)15702-6]

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The growing interest in the study of time-modulated free convection, started first by Gershuni and Zhukhovitsky [1], is triggered by unfolding new mechanisms of instability. In such systems as binary liquids [2], multilayer systems [3], and nematic liquid crystals [4], which undergo Hopf bifurcation in the absence of modulation, strong destabilization based on the resonance phenomena can be observed in the presence of modulation [5–7].

It seems that less attention was paid to the fact that in an extended system with translational and reflection symmetries undergoing Hopf bifurcation, when modulation amplitude $\hat{\eta}$ is small and the group velocity of an oscillatory mode does not vanish, the resonance-instability domain expands uniformly when $\hat{\eta}$ increases. This results in a new set of amplitude equations, which due to nontraditional but relevant scaling contains advective terms but leaves out the diffusive ones. In this paper we derive these equations by performing a weakly nonlinear analysis of Soret-driven convection in a binary mixture, although they may be encountered in a wider class of extended systems with the above-mentioned symmetries.

Consider an infinite horizontal layer of incompressible binary mixture heated from below and subject to vertical vibrations. The basic equations are the Boussinesq equations for a binary mixture with Soret effect and modulated body force, normalized by the layer's depth and viscous time for space and time, respectively,

$$(\nabla^4-\partial_t\nabla^2)\psi+RP^{-1}(1+\hat{\eta}\sin\omega t)\partial_x(T+\Psi C)=N(\nabla^2\psi),$$

$$(P^{-1}\nabla^2-\partial_t)T+\partial_x\psi=N(T),$$

$$-\partial_t C+LP^{-1}\nabla^2(T+C)+\partial_x\psi=N(C).$$

Here ψ is the stream function; T and C are deviations of temperature and concentration from their linear equilibrium profiles, respectively; ∂_x,∂_t denote derivatives with respect to the corresponding variable, $N(f)\equiv\partial_x\psi\partial_z f-\partial_z\psi\partial_x f$; R is the Rayleigh number; ω and $\hat{\eta}$ are frequency and relative amplitude of modulation, respectively; P is the Prandtl number; L is the Lewis number; and Ψ is the separation ratio (Soret parameter).

Earlier, the linear stability analysis of the equilibrium state $\psi=T=C=0$ was carried out in [5,6]. Saunders *et al.* [5] considered simplified free-free boundary conditions $\psi=\partial_z^2\psi=T=C=0$ at $z=0,1$. A more realistic configuration was considered by Terrones *et al.* [6] for quite a narrow range of parameters, since the variety of mechanisms included in the problem (Dufour and Soret effects, an imposed solutal gradient) makes a comprehensive study almost impossible. They revealed a strong destabilizing effect of modulation on stability of the equilibrium for negative Ψ, finite $\hat{\eta}$, and resonant frequencies.

Let us first briefly present the results of the linear stability analysis of the equilibrium. The boundary conditions $\psi=\partial_z\psi=T=\partial_z(C+T)=0$ at $z=0,1$ are valid for rigid impermeable boundaries held at uniform temperature. The numerical procedure is similar to that used in [6]: the basic equations are projected onto a finite-dimension space using the Galerkin method and the resulting set of the ODE's is analyzed using the Floquet technique. A typical stability map displays a set of neutral curves $R_c(k)$ for various values of $\hat{\eta}$. In the unmodulated case, $\hat{\eta}=0$, the neutral curve is a manifold of Hopf bifurcation points (Fig. 1).

When the modulation is turned on, the modes whose Hopf frequency is resonant to the modulation frequency become strongly destabilized. In Fig. 1 such a resonance mode has $k=3.12$ and its Hopf frequency σ is $\omega/2$ (so-called strong resonance).

A frequency locking and destabilization of neighboring modes leads to the emergence of resonance instability domains confined by curve 1 and curves 2–5, respectively. These domains are geometrically similar, and in the strong-resonance case they expand proportionally to $\hat{\eta}$. Our calculations carried out for different frequencies show that secondary resonances, with $m\omega=2\sigma$, $m=2,3\ldots$, are also possible, but all these have a weak influence on the stability threshold as the corresponding instability domains expand as $\hat{\eta}^m$.

Let us now turn to the weakly nonlinear analysis inside the resonance-instability domain. We introduce a small parameter μ, measuring subcriticality, and different time and space scales, t,τ,x,X. In the vicinity of the resonant frequency, i.e., when $m\omega=2\sigma+O(\mu)$, one has $\Delta R\sim\Delta k\sim\hat{\eta}^m\sim\mu$, thus, the relevant scaling for the ''slow''

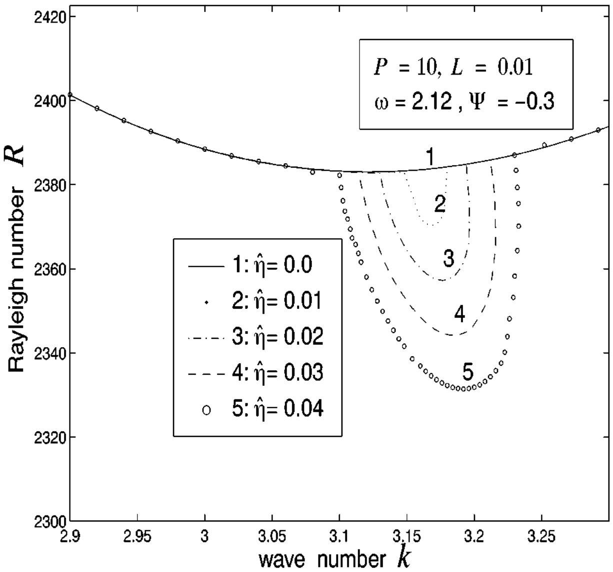

FIG. 1. A typical stability map of the equilibrium showing the Rayleigh number R_c vs wave number k. Curve 1 shows the neutral curve in an unmodulated case. Curves 2–5 correspond to $\hat{\eta}\neq 0$. Parameters shown are relevant for 0.4% wt. water-ethanol mixture at 15 °C [8]. If modulation is produced by vertical vibration of a layer of the depth of 1.5 mm on Earth, then the dimensional amplitude and frequency of vibrations are $\simeq$50 cm and $\simeq$1 sec^{-1}, respectively, for $\hat{\eta}=0.05$.

time and space variables is $\tau=\mu t$, $X=\mu x$. A state of the system is described by superposition of two counterpropagating waves

$$\mathbf{U}=a_1\mathbf{u}_1(z)e^{i(kx-\sigma t)}+a_2\mathbf{u}_2(z)e^{i(kx+\sigma t)}+\text{c.c.}$$

Here vector $\mathbf{U}=\{\psi,T,C\}$ describes the state of the system, $a_1(\tau,X),a_2(\tau,X)$ are complex amplitudes slowly varying with time and space, and $\mathbf{u}_1,\mathbf{u}_2$ are vertical eigenvectors of the corresponding linear eigenvalue problem at the Hopf bifurcation point.

The coupled equations for a_1 and a_2 are obtained from the solvability condition at order $\mu^{3/2}$. At this order one obtains after a proper rescaling the following set:

$$\partial_\tau a_1-\partial_X a_1=(\lambda_1|a_1|^2+\lambda_2|a_2|^2+\rho)a_1+\eta a_2, \quad (1a)$$

$$\partial_\tau a_2+\partial_X a_2=(\lambda_2^*|a_1|^2+\lambda_1^*|a_2|^2+\rho^*)a_2+\eta a_1. \quad (1b)$$

Here λ_1,λ_2 are complex coefficients, $\lambda_{1r}=\pm 1$; indices "r" and "i" from now on denote real and imaginary parts, respectively; $\eta\sim\hat{\eta}^m$ is a real positive parameter representing the rescaled amplitude of modulation; $\rho=\alpha i-1$ is a linear gain and α is the detuning parameter.

The negative real part of ρ corresponds to the subcritical domain with respect to the unmodulated case. In the supercritical domain, i.e., above the neutral curve 1 in Fig. 1, the bandwidth of unstable modes is $\Delta k\sim\mu^{1/2}$, which calls for two slow space scales, $X_1=\mu^{1/2}x$, $X_2=\mu x$. From the solvability condition at order μ one then obtains two equations for a_1 and a_2 with advective terms $s\partial_{X_1}a_1$ and $-s\partial_{X_1}a_2$, where s is the group velocity [7,9]. These terms cannot be eliminated by using a moving frame of reference like in the case of a single wave, because of the cross terms, describing the waves' interactions. To overcome this difficulty several authors assumed smallness of s [9], used averaging over the scale X_1 [10], or focused on a single wave, neglecting the second one [11].

The coefficients of Eqs. (1) for Soret-driven convection were calculated in [12]. The group velocity s was found to be finite, thus omitting the advective terms or pushing them into the next order (that allows the derivation of the amplitude equation) yields incorrect equations in the unmodulated case. In the case of (resonant) modulation these terms take their place in the amplitude equation naturally, due to the appropriate scaling.

Consider monochromatic (MC) solutions for Eqs. (1) in the form $a_j=A_j\exp(iKX+i\theta_j)$, $j=1,2$, where amplitudes A_j and phases θ_j are real functions of τ. One readily obtains a set of three real equations for A_1,A_2 and $\theta\equiv\theta_1-\theta_2$ (due to the translational symmetry $\theta_1+\theta_2$ decouples from the equations; see [14]):

$$\partial_\tau A_1=\rho_r A_1+\eta A_2\cos\theta+\lambda_{1r}A_1^3+\lambda_{2r}A_2^2A_1,$$

$$\partial_\tau A_2=\rho_r A_2+\eta A_1\cos\theta+\lambda_{1r}A_2^3+\lambda_{2r}A_1^2A_2, \quad (2)$$

$$\partial_\tau\theta=\left(\lambda_{1i}+\lambda_{2i}-\frac{\eta}{A_1A_2}\sin\theta\right)(A_1^2+A_2^2)+2\delta,$$

where δ is a generalized detuning coefficient, $\delta=K+\alpha$.

Equations (2) were considered earlier by Riecke *et al.* [14] and Walgraef [13] mainly in the supercritical (with respect to unforced system) domain, i.e., for $\rho_r=1$. Our results, corresponding to the case $\rho_r=-1$, are in qualitative agreement with those received by [14].

Equations (2) admit two kinds of stationary nontrivial solutions. The solution of the first kind, with $A_1=A_2=\mathcal{A}$, $\theta_1=-\theta_2=\Theta$, describes standing waves (SW),

$$\mathcal{A}_\pm^2=[n-\delta\pm\sqrt{\eta^2(n^2+1)-(1+\delta n)^2}]/\lambda_i(1+n^2),$$

$$\eta\sin\Theta_\pm=\lambda_i\mathcal{A}_\pm^2+\delta,\quad \lambda=\lambda_1+\lambda_2,\quad n=\lambda_r/\lambda_i.$$

The stability of SW with respect to harmonic perturbations (of the same wavelength as SW) can be studied in the framework of Eqs. (2).

SW branches off the trivial state of Eqs. (2) at the stationary bifurcation point $\eta_S^2=1+\delta^2$ supercritically (subcritically) [14] when $n-\delta<0(>0)$. The most unstable mode losing stability first has the wave number $K=-\alpha$. In the case of subcritical bifurcation there is the saddle-node bifurcation point $\eta_{SN}=(1+n\delta)/\sqrt{1+n^2}$, where two branches $\mathcal{A}_+$ (stable) and $\mathcal{A}_-$ (unstable) merge.

In the case $n>0$ with a further increase of the modulation amplitude η, SW loses stability either with respect to oscillatory perturbation of SW-type at the Hopf bifurcation point $\eta_H=\sqrt{1/4+(1/2n-\delta)^2}$, where stable periodic SW bifurcates supercritically, or with respect to perturbations of a traveling wave's type at $\eta_T=\sqrt{(\lambda_{2r}-1)^2+(\lambda_i+2\delta)^2}/2$, where the stationary traveling wave bifurcates subcritically.

Stability of periodic SW, corresponding to two-frequency wave motion (waves with periodically time-modulated am-

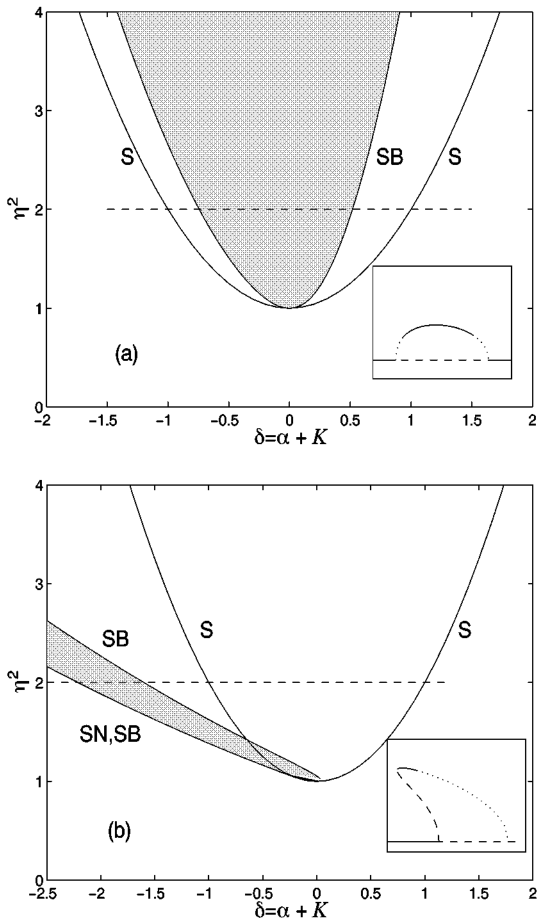

FIG. 2. The phase diagram of a MC SW for (a) $n=-3$ and (b) $n=-0.2$. The curve SN consists of saddle-node bifurcation points; the curve S shows the linear stability threshold of the equilibrium and consists of steady bifurcation points; shaded is the domain of SB-stability of MC SW bounded by the curve SB. Insets: Bifurcation diagram of MC SW obtained along the corresponding dashed line. The stable branch is shown by a solid curve, the unstable ones branching subcritically by a dashed curve, and the SB-unstable one by a dotted curve.

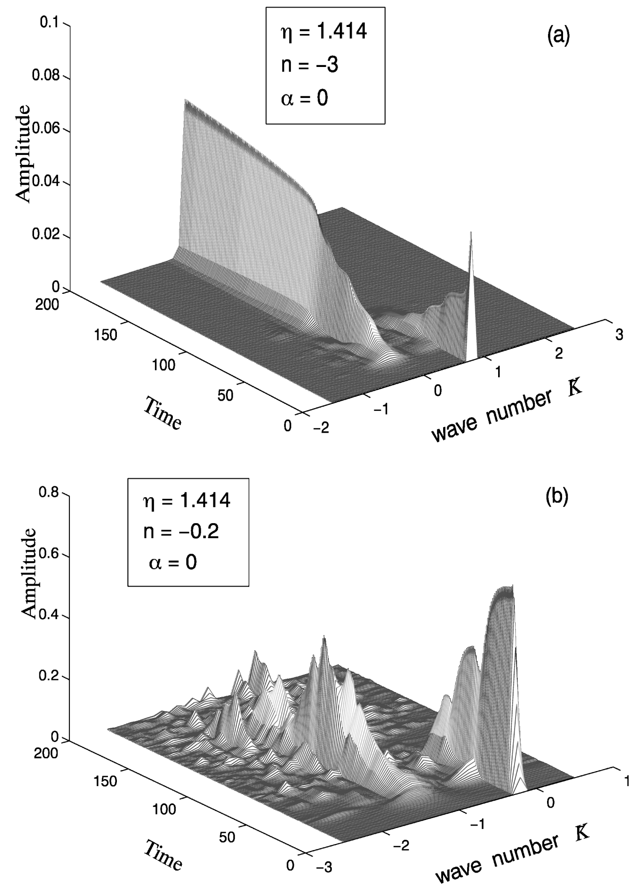

FIG. 3. The time evolution of the Fourier spectrum of the standing wave's envelope for (a) $n=-3$ and (b) $n=-0.2$. As an initial condition we use a MC SW corresponding to $\widetilde{K}=0.8$, $\alpha=0$ and $\widetilde{K}=-0.7$, $\alpha=0$, respectively, perturbed by a weak noise. Their corresponding stationary states are SB-unstable and therefore collapse. As a result, a different SB-stable stationary SW (a) or an irregular regime (b) sets in.

plitudes), is next studied numerically. It loses stability while merging with unstable solutions $\mathcal{A}_-$ or origin $A=0$, producing a homoclinic orbit. At larger η no confined stable solutions are found; all trajectories extend to infinity.

The stationary solution of the second kind, traveling wave (TW) with $A_1 \neq A_2 \neq 0$, describes two counterpropagating waves with different amplitudes and phase velocities [14]. In our case of a subcritical (with respect to an unforced system) regime, TW is unstable; trajectories starting near TW either approach the stable (stationary or periodic) SW or extend to infinity. In the supercritical case, $\rho_r=1$, as reported in [14], TW can be stable.

We now present the results of the stability analysis of stationary MC SW with respect to sideband (SB) perturbations. This is done in the framework of Eqs. (1). For simplicity we consider here the case $\lambda_2 \to 0$, which corresponds to the weak nonlinear interaction of the envelopes in comparison with saturation. We introduce a transformation $S(f(X,\tau))=f^*(-X,\tau)$ describing the SW symmetry and through which the amplitudes a_1, a_2 are interrelated (in the case of SW). This allows us to study the envelope of a nonmonochromatic time-dependent SW using only one equation for propagating, say, rightward component. Substituting $\lambda_2=0$, $a_1=\lambda_i^{-1/2}b$ into Eq. (1a) yields

$$\partial_\tau b - \partial_X b = (i\alpha - 1)b + \eta S(b) + (n+i)|b|^2 b. \tag{3}$$

We consider the SW solution perturbed by the wave with different wave numbers $K+q$ and $K-q$:

$$b = e^{iKX + i\Theta}(\sqrt{\lambda_i}\mathcal{A} + \beta_1 e^{-iqX} + \beta_2 e^{iqX}). \tag{4}$$

Substituting Eq. (4) into Eq. (3) yields the dispersion relation, which reveals that the long-wave perturbation is the most dangerous one. Even in the long-wave limit $q \to 0$ the dispersion relation has a complicated implicit form and is analyzed numerically.

We find that for $n>0$ all MC SW are SB-unstable. This could be expected, since in this case the cubic term in Eq. (3) does not saturate the instability. For slightly negative n the domain of SB-stability is located near the left boundary of the SW's existence domain and broadens when n decreases. For $n \lesssim -1$, the subdomain of SB-stability is surrounded by

SB-instability regions. It moves towards the center of existence domain and both become more symmetric with respect to the vertical axis $\delta=0$.

The phase diagrams for $n=-3,-0.2$ are shown in Fig. 2. In the case $n=-3$ all solutions (except for those with $\delta<n$) bifurcate supercritically (with η taken as a bifurcation parameter) and the SB-stable SW is not separated from the origin by an unstable manifold. Therefore, any small MC-perturbation grows to a stationary SW, if the latter is SB-stable.

In the case $n=-0.2$, all MC SW bifurcating supercritically are SB-unstable for almost any η. For $\eta>1.2$, SB-stable MC SW's are possible only below the linear stability threshold η_S, and they are separated from the stable zero solution by the unstable branch. Therefore, stationary MC SW can be reached only via a finite-amplitude perturbation. For small amplitudes $\eta<1.2$ there exists a bandwidth of SB-stable SW's which are not separated from the origin by an unstable manifold and can be reached from an infinitesimal perturbation.

We solve Eq. (3) numerically to verify and extend the results of our linear stability analysis and to study the behavior of nonmonochromatic waves. Equation (3) is solved using the second-order predictor-corrector scheme in $0\leqslant X\leqslant 20\pi$ with the periodic boundary conditions. The accuracy of the numerics is checked by using different time and space discretization steps. Moreover, when a stationary SW solution is achieved, its amplitude and phase are in perfect agreement with those obtained analytically.

The numerical simulation of Eq. (3) confirms the results of the linear stability analysis. Figure 3 shows the evolution of the SW's spectra for $\eta=\sqrt{2}$ and $n=-3,-0.2$. As an initial condition we use a MC-wave with a weak noise within the band shown in Fig. 2 by dashed lines.

In the case $n=-3$ [see Fig. 3(a)], the initial MC-perturbation corresponds to $\delta=0.8$ and it is SB-unstable in agreement with the linear stability analysis. This SW first approaches its stationary state, which collapses, transferring its energy via the nonlinear term to other harmonics. One of them, with $\delta=-0.1$, achieves its stationary state, which according to the linear stability analysis is SB-stable.

In the case $n=-0.2$ [see Fig. 3(b)], the MC-perturbation initially also tends to a stationary state, which being SB-unstable collapses. However, none of the harmonics achieves its final amplitude, since it does not possess enough energy to surpass the barrier—an unstable manifold [see inset in Fig. 2(b)]. If it does, it would achieve a stationary state and would be SB-stable. As a result of the SB-instability, an irregular behavior sets in. Numerical calculations show that it persists with unfading spectral amplitudes during more than 2000 units of dimensionless time. To our knowledge, the chaos-driving mechanism here differs from those in [7,15] and is based on the following factors: (i) SB-stable SW's are separated from zero solution by an unstable manifold; (ii) other SW's are SB-unstable; (iii) the zero solution is unstable; (iv) the energy of initial perturbation is sufficiently low.

As a conclusion, we derived a set of amplitude equations for the envelope of waves excited by the parametric resonance. The sideband instability of SW was found to be a route for the onset of an irregular behavior for a small saturation coefficient. For a strong nonlinear damping the sideband instability of a MC-wave leads to the emergence of a MC-wave with a different wavelength.

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